Geometry question.

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Consider a triangle ABC, where $AB = 20, BC = 25$ and $CA = 17$. P is point on the plane. What is minimal value of $2 \cdot PA + 3 \cdot PB + 5 \cdot PC$?

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Let $\overline{PA}, \overline{PB}, \overline{PC}$ be directed line segments, with lengths $|PA|, |PB|, |PC|$, respectively

Then $F(P) := 2|\overline{PA}| + 3|\overline{PB}| + 5|\overline{PC}| = 2(|\overline{PA}| + |\overline{PC}|) + 3(|\overline{PB}| + |\overline{PC}|) =$
$2(|\overline{AP}| + |\overline{PC}|) + 3(|\overline{BP}| + |\overline{PC}|) \geq 2|\overline{AP} + \overline{PC}| + 3(|\overline{BP} + \overline{PC}|) = 2|\overline{AC}| + 3|\overline{BC}| = 2 \cdot 17 + 3 \cdot 25 = 109$.

Thus $\min F(P) = 109 = F(C)$.

Generalization.

Let $A_1, A_2, \ldots, A_n$ be n points on a plane $\mathcal{P}$. And let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be positive numbers such that $\alpha_1 \geq \alpha_2 + \ldots + \alpha_n$. Find $\min\limits_{P \in \mathcal{P}}(\alpha_1|PA_1| + \alpha_2|PA_2| + \ldots + a_n|PA_n|)$.

(Here PM is directed from P to M line segment with length $|PM|$)

Solution.

$F(P) := \sum_{i=1}^{n} \alpha_i|PA_i| = \left(\alpha_1 - \sum_{i=2}^{n} \alpha_i\right)|PA_1| + \sum_{i=2}^{n} \alpha_i(|PA_i| + |PA_1|) \geq$
$\sum_{i=2}^{n} \alpha_i(|A_1P| + |PA_i|) \geq \sum_{i=2}^{n} \alpha_i|A_1P + PA_i| = \sum_{i=2}^{n} \alpha_i|A_1A_i| = F(A_1)$.

Or, in vector form:

Let $\mathbf{r} := \overline{OP}$ and $\mathbf{a}_i := \overline{OA_i}, i = 1, 2, \ldots, n$. Then $\sum_{k=1}^{n} \alpha_i|PA_i| = F(\mathbf{r}) := \sum_{k=1}^{n} \alpha_i|\mathbf{r} - \mathbf{a}_i| =$
$\left(\alpha_1 - \sum_{k=2}^{n} \alpha_i\right)|\mathbf{r} - \mathbf{a}_1| + \sum_{k=2}^{n} \alpha_i(|\mathbf{r} - \mathbf{a}_i| + |\mathbf{r} - \mathbf{a}_1|) \geq \sum_{k=2}^{n} \alpha_i(|\mathbf{r} - \mathbf{a}_i| + |\mathbf{a}_1 - \mathbf{r}|) \geq \sum_{k=2}^{n} \alpha_i|\mathbf{r} - \mathbf{a}_i + \mathbf{a}_1 - \mathbf{r}| =$
$\sum_{k=2}^{n} \alpha_i|\mathbf{a}_1 -- \mathbf{a_i}| = F(\mathbf{a}_1)$. Thus, $\min\limits_{\mathbf{r}} F(\mathbf{r}) = F(\mathbf{a}_1)$.

Remark.

If inequality $\alpha_1 \geq \alpha_2 + \ldots + \alpha_n$ ($\alpha_1 = \max\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$) isn't holds then minimum can't be attained in no one of the points $A_1, A_2, \ldots, A_n$

As example: $\min(|\overline{PA}| + |\overline{PB}| + |\overline{PC}|) = |\overline{TA}| + |\overline{TB}| + |\overline{TC}|$ where T is Fermat–Torricelli point.